Integrated Analysis of Computer and Physical Experiments

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Abstract

In scientific investigations, we frequently have data from computer experiment(s) as well as related physical experimental data on the same factors and related response variable(s). There may also exist one or more expert opinions regarding the response of interest. Traditional statistical approaches consider each of these sets of data separately with corresponding separate analyses and fitted statistical models. A compelling argument can be made that better, more precise statistical models can be obtained if we simultaneously analyze the combined data using a hierarchical Bayesian integrated modeling approach. However, such an integrated approach must recognize important differences, such as possible biases, in these experiments and expert opinions.

We illustrate the methodology by using it to model the thermodynamic operation point of a top-spray fluidized bed microencapsulation processing unit. Such units are used in the food industry to tune the effect of functional ingredients and additives. An important thermodynamic response variable of interest, Y, is the steady-state outlet air temperature. In addition to a set of physical experimental observations involving six factors used to predict Y, similar results from three different computer models were also available. The integrated data from the physical experiment and the three computer models are used to fit an appropriate response surface (regression) model for predicting Y.

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1 Introduction

Computer models are often used to perform experiments before expensive physical experiments are performed. The computer models attempt to reproduce the physical properties of a process by mathematically representing the individual physical sub-processes. For example, in the food industry, fluidized bed (or air suspension) processes are increasingly used to coat food particles with preservatives and flavor enhancers. Some of the physical principles that govern the operation of fluidized beds are fairly well-understood (e.g., heat transfer and fluid flow), but others are less well-characterized. As a result, computer models, based on these thermodynamic principles of physics, are constructed that resemble and simulate the actual physical process. In this paper we analyze data collected from three such computer models (each accounting for different effects) as well as data collected from a corresponding physical experiment. We consider this example further in Section 3.

It is statistically efficient and desirable to fit a single, common response surface model that combines the physical experimental data and the computer model output data to express the relationship between the factors and the response variable. Although the response variables of interest in the computer and physical experiments may not be the same, we assume that they can be related by a known transfer function. Thus, we effectively consider the same response variable in both types of experiments. However, the computed (or measured) value of the response variable need not be considered at the same factor values in both experiments. We require only that there exist some common set of factors (either all or at least some) for both experiments (see Section 2.3). For example, a broad (screening) computer experiment may be performed first, followed later by a physical experiment in a smaller region of particular interest (perhaps a corner) of the overall computer experiment design space.

In addition, one or more expert opinions may be available regarding the response variable of interest. Traditional statistical approaches consider each of these sets of data separately with corresponding separate designs, analyses, and results. A compelling argument can be made that better, more powerful statistical results can be obtained if we simultaneously analyze the

combined data using a recursive Bayesian hierarchical model (RBHM) that we propose in Section 2. As we will illustrate, the simultaneous analysis of such combined data permits the unknown coefficients in an assumed overall regression (or response surface) model to be estimated more precisely, thereby producing a better fitting response surface.

In Section 2 we present the methodology including our implementation of the RBHM. Section 3 contains a description of the mechanics and process variables involved in the fluidized bed example and the experiment from which the data arise. We apply the RBHM methodology to the fluidized bed study and present the resulting response surface in Section 3.3. Sensitivity to prior specification is studied in Section 4. Finally, in Section 5 we discuss the results and methodology.

2 Data Integration Model and Analysis

Fundamental to Bayesian estimation is the notion and use of prior and posterior distributions. A good elementary discussion of prior and posterior probabilities and distributions is given in Berry (1996). An RBHM provides a convenient way to sequentially combine the data as follows. A set of initial informative, but diffuse, prior distributions is defined, one for each unknown parameter. If they exist, any available expert opinion data are then used to update these priors to form corresponding posterior distributions. This represents Stage 1 of the combined analysis. These posteriors then become the prior distributions for the second stage, in which the computer experimental data are used to update these priors to form Stage 2 posterior distributions. At Stage 2, the posteriors thus represent the combined use of only the expert opinion and computer data. Finally, these posteriors become the priors for Stage 3, in which the physical experimental data are used to construct the final desired posteriors. In this way, all available data are used recursively within the context of the model to successively (and more precisely) estimate all the desired parameters of interest.

The design and analysis of computer experiments has evolved as the power of computers has grown (although it has certainly not kept pace!). Sacks, et al. (1989) provide a review

of techniques used in the analysis of output from complex computer codes as well as issues for design. Latin hypercube sampling had its genesis in the design of computer experiments (McKay, Beckman, and Conover 1979). A Bayesian treatment of the design and analysis of computer experiments is presented in Currin et al. (1991). These papers are concerned primarily with issues when the only source of information is the output from a complex computer model.

Combining multiple sources of information had its genesis in the meta analytic literature. Zeckhauser (1971) provides an early treatment of meta analysis. Hedges and Olkin (1987) provide a nice review of meta analytic techniques. Meta analysis has not been viewed without strong criticism (Shapiro 1994 and discussion). Müller et al. (1999) present a Bayesian hierarchical modeling approach for combining case-control and prospective studies, where effects due to different studies as well as different centers are allowed.

Craig, et al. (2001) present an approach to forecasting from computer models which explicitly incorporates two of the data sources we consider, expert opinion and computer experiments. Considered there is the possibility of multivariate responses on the computer model (which they call computer simulators). Physical data in the form of historical measurements is included by using this information in prior (expert opinion specification). Their approach is based on a Bayesian treatment with no hierarchical modeling and inventive ways of including several types of expert opinion. The primary concern is with improvement of the prediction of the computer code.

Kennedy and O'Hagan (2001) consider the three sources of data that we consider here in this paper. Their approach uses a general Gaussian process model for the computer model as a function of inputs. They use physical data to calibrate the computer experimental data and to estimate unknown parameters of that model. They also find Bayesian hierarchical models a useful tool in implementation of their models. Their framework is flexible and, in the context of trying to improve computer models, the appropriate approach. The essential difference between their work and our proposal is that we are trying to use computer model outputs and expert opinion to improve estimation and prediction of the physical process, and Kennedy and O'Hagan

are trying to use physical experimental data and expert opinion to improve the computer model.

The statistical notion of pooling data (sometimes also known as "borrowing strength") underlies the RBHM and analysis to be discussed. A commonly used, and extremely powerful method for borrowing strength is hierarchical Bayesian modeling. A nice introduction to both hierarchical Bayesian modeling and borrowing strength is given in Draper et al. (1992). The basic idea involves the notion that, when information concerning some response of interest arises from several independent, but not identical, data sources, a hierarchical model is often useful to describe relationships involving the observed data and unobserved parameters of interest. For example, unobserved parameters might be the coefficients and error variance in an assumed response surface model, as well as unknown biases. Each source of data provides perhaps biased information about these parameters, in which case methods that borrow strength will be useful. The practical advantages of borrowing strength for estimating the unknown parameters will be illustrated in Section 3.2.

We propose fitting models using information from three distinct sources: expert opinion, computer experiments, and physical experiments. The problem is difficult because the information sources are not necessarily all available at each of the design points. For example, physical experiments may be performed according to a statistically designed experiment, while computer experiments may be collected at (possibly) different design points. In addition, expert opinions may be available at only a very limited set of design points, such as the center or corners of the statistical design region. Our goal is to combine these sources of information using an appropriately flexible integration methodology that considers (and automatically adjusts for) the uncertainties and possible biases in each of these three data sources.

Thus, we begin by considering regression models of the form:

$$\underline{Y} = f(\mathbf{X}, \underline{\beta}) + \underline{\varepsilon},$$

where $\mathbf{X}$ is a design matrix, $\underline{\beta}$ is a vector of unknown coefficients, and $\underline{\varepsilon}$ is a vector of unobserved errors. Note that while this formulation can accommodate a general class of models, $f(\cdot)$, including both linear and nonlinear regression models, in this and the following sections we

consider only linear models (that is, $f(\mathbf{X}, \underline{\beta}) = \mathbf{X}\underline{\beta}$). While the strategy we employ is quite general, the model and mathematics we develop in the following sections is applied to a normal linear model. In addition, we consider only quantitative variables, although qualitative variables coded with indicator variables fit naturally into this framework.

2.1 Physical Experimental Data

We assume we are interested in estimating the parameters of a model that describes a physical experiment. For this example, assume that the physical experimental data can be described using the following familiar model:

$$\underline{Y}_p \sim N(\mathbf{X}_p\underline{\beta}, \sigma^2 I),$$

where the subscript p denotes the "physical experiment." Thus, the physical experimental data are assumed to be normally distributed with mean $\mathbf{X}_p\underline{\beta}$, where $\mathbf{X}_p$ is a model matrix and $\underline{\beta}$ is a vector of parameters that need to be estimated. We see that each physical observation is independent of the others and has common (homoscedastic) variance σ^2, which must also be estimated.

If physical experimental data were the only information source considered, this model would typically be fit using either standard least-squares regression methods (Draper and Smith 1981) or standard Bayesian linear model methods (Gelman et al. 1995). However, we want to incorporate information both from experts and computer experimental data to "improve" our estimates of $\underline{\beta}$ and σ^2.

2.2 Expert Opinion

Suppose there are e expert opinions. These opinions do not have to be from distinct experts. The i^{th} expert opinion ($i = 1, \ldots, e$) is elicited at design point x_i. Some points in the design space will have exactly one elicited expert opinion; others will have many or none. Each expert observation contains the following information:

- the expected response, y_{o_i}

- a subjective coverage probability on the physical response y_i, ξ_i, and the quantile associated with that probability, q_{ξ_i} (i.e., $Pr(y_i \leq q_{\xi_i}) = \xi_i$)

Typically, the analyst elicits a quantile of interest, that is, ξ_i is specified. However, the expert may indicate which quantile she is most interested in specifying. The methods developed here do not depend on which approach is taken. In addition, we consider the elicited "worth" of the opinion in units of equivalent physical experimental data observations, $m_{o_i}^{(e)}$. In other words, suppose that a physical experiment could be conducted at x_i that would yield one observation: if the expert's opinion should be weighted half of that observation then $m_{o_i}^{(e)} = 0.5$. At times, the elicited values (y_{o_i}, ξ_i, q_{ξ_i}, $m_{o_i}^{(e)}$) may be obtained simply by requesting them from the expert. However, it may be difficult for the expert to provide information directly on these values (especially q_{ξ_i} and $m_{o_i}^{(e)}$), and other elicitation techniques may be useful (Meyer and Booker 1990).

In order to use these data, we need to transform these individual pieces of information into probability distributions that provide information about $\underline{\beta}$ and σ^2. Assume for the moment that the three quantities above can be used to create "data" with the following model:

$$\underline{Y}_o \sim N(\mathbf{X}_o\underline{\beta} + \underline{\delta}_o, \sigma^2\mathbf{\Sigma}_o).$$

As with the physical experimental data, the expert data are assumed to be normally distributed. However, the mean is $\mathbf{X}_o\underline{\beta} + \underline{\delta}_o$, where $\underline{\delta}_o$ is a vector of location biases that are expert specific. The variances are also biased, and the matrix $\mathbf{\Sigma}_o$ contains the scale biases for each expert. Besides location biases, in which an expert's average value is high or low relative to the true mean, scale biases often occur due to information over-valuation and are well-documented in the elicitation literature. For example, an expert may be asked to provide what they think is a 0.90 quantile, but responds with what is actually only a 0.60 quantile (Meyer and Booker 1990). Although responses from experts can be correlated by having non-diagonal elements in

$\mathbf{\Sigma}_o$, we consider uncorrelated responses; thus,

$$\mathbf{\Sigma}_o = \begin{bmatrix} 1/k_{o_1} & 0 & \cdots & 0 \\ 0 & 1/k_{o_2} & 0 & \cdots \\ \vdots & 0 & \ddots & \cdots \\ 0 & \cdots & \cdots & 1/k_{o_e} \end{bmatrix}.$$

In addition, assume the following prior distributions for the unknown parameters $\underline{\beta}$ and σ^2:

$$\begin{aligned} \underline{\beta}|\sigma^2 &\sim N(\underline{\mu}_o, \sigma^2 \mathbf{C}_o) \\ \sigma^2 &\sim IG(\alpha_o, \gamma_o), \end{aligned}$$

where $IG(a, b)$ is the inverse gamma distribution with density function

$$f(z|a,b) \propto z^{-(a+1)} \exp\left\{-\frac{b}{z}\right\}, \qquad z > 0.$$

Assume for the moment that we know $\underline{\delta}_o$ and $\underline{m}_o$, where $\underline{m}_o$ is a vector denoting the "worth" of the expert opinions. Continue to assume that we have created "data" $\underline{y}_o$ from the expert opinions, and write out the likelihood for the data model:

$$\left(\frac{1}{|\sigma^2 \mathbf{\Sigma}_o|^{.5}}\right) \exp\left\{-\frac{1}{2\sigma^2}[(\underline{y}_o - (\mathbf{X}_o\underline{\beta} + \underline{\delta}_o))' \mathbf{\Sigma}_o^{-1} (\underline{y}_o - (\mathbf{X}_o\underline{\beta} + \underline{\delta}_o))]\right\}.$$

Using Bayes' Theorem, we can use the data provided by the expert opinions to update the prior distributions for $\underline{\beta}$ and σ^2. The resulting Stage 1 posterior/updated prior distribution for $(\underline{\beta}, \sigma^2)$, conditional on $\underline{\eta} = (\underline{\delta}_o, \mathbf{\Sigma}_o, \underline{m}_o, \mathbf{C}_o, \underline{\mu}_o, \alpha_o, \gamma_o)$, is

$$\begin{aligned} \pi(\underline{\beta}|\sigma^2, \underline{\eta}, \underline{y}_o) &\sim N\left((\mathbf{X}_o' \mathbf{\Sigma}_o^{-1} \mathbf{X}_o + \mathbf{C}_o^{-1})^{-1} \underline{z}, \sigma^2 (\mathbf{X}_o' \mathbf{\Sigma}_o^{-1} \mathbf{X}_o + \mathbf{C}_o^{-1})^{-1}\right) \\ \pi(\sigma^2|\underline{\eta}, \underline{y}_o) &\sim IG\left(\alpha_o + \frac{\sum_{i=1}^{e} m_{o_i}}{2}, \right. \\ &\quad \gamma_o + .5[(\underline{y}_o - \underline{\delta}_o)' \mathbf{\Sigma}_o^{-1} (\underline{y}_o - \underline{\delta}_o) + \underline{\mu}_o' \mathbf{C}_o^{-1} \underline{\mu}_o - \\ &\quad \left. \underline{z}' (\mathbf{X}_o' \mathbf{\Sigma}_o^{-1} \mathbf{X}_o + \mathbf{C}_o^{-1})^{-1} \underline{z}]\right), \end{aligned}$$

where $\underline{z} = \mathbf{X}_o' \mathbf{\Sigma}_o^{-1} (\underline{y}_o - \underline{\delta}_o) + \mathbf{C}_o^{-1} \underline{\mu}_o$.

Given that the full vector of observations $\underline{y}_o$ was not elicited (only sufficient statistics were), we cannot immediately evaluate any term in these expressions. We re-express the components

in these posterior distributions in terms of the elicited values instead, so they can be evaluated. Suppose m_{o_i} observations were elicited as $\underline{y}_{o_i}$ from the i^{th} expert opinion. Then

$$\begin{aligned}(\mathbf{X}_o'\mathbf{\Sigma}_o^{-1}(\underline{y}_o - \underline{\delta}_o))_j &= k_{o_1}x_{1j}\left(\sum_{n=1}^{m_{o_1}}(y_{o_{jn}} - \delta_{o_1})\right) + \ldots + k_{o_e}x_{ej}\left(\sum_{n=1}^{m_{o_e}}(y_{o_{jn}} - \delta_{o_e})\right) \\ &= k_{o_1}m_{o_1}x_{1j}(y_{o_1} - \delta_{o_1}) + \ldots + k_{o_e}m_{o_e}x_{ej}(y_{o_e} - \delta_{o_e}),\end{aligned}$$

as y_{o_i} is the expected or average response for the design point.

Using a similar argument, we can show that

$$\begin{aligned}(\underline{y}_o - \underline{\delta}_o)'\mathbf{\Sigma}_o^{-1}(\underline{y}_o - \underline{\delta}_o) &= k_{o_1}\left(\sum_{n=1}^{m_{o_1}}(y_{o_{jn}} - \delta_{o_1})^2\right) + \ldots + k_{o_e}\left(\sum_{n=1}^{m_{o_e}}(y_{o_{jn}} - \delta_{o_e})^2\right) \\ &= \sum_{i=1}^{e} k_{o_i}m_{o_i}(s_i^2 + (y_{o_i} - \delta_{o_i})^2), \qquad (1)\end{aligned}$$

where $s_i^2 = (y_{o_i} - q_{\xi_i})^2/Z_\xi^2$, which is the variance approximation implicitly elicited from expert i. Equation (1) follows from the identity $Var(Y) = E[Y^2] - E[Y]^2$.

By a similar argument

$$(\mathbf{X}_o'\mathbf{\Sigma}_o^{-1}\mathbf{X}_o)_{ij} = \sum_{n=1}^{e} k_n m_{o_n} x_{ni}x_{nj}.$$

These representations allow the quantities in the posterior distributions to be calculated based on the elicited values rather than the actual observations.

For the unknown parameters $\underline{\eta} = (\underline{\delta}_o, \mathbf{\Sigma}_o, \underline{m}_o, \mathbf{C}_o, \underline{\mu}_o, \alpha_o, \gamma_o)$ we propose the following prior distributions:

$$\begin{aligned}
\underline{\mu}_o &= \underline{a}_{\mu_o} \\
\mathbf{C}_o &= a_{C_o}\mathcal{I} \\
\alpha_o &= a_{\alpha_o} \\
\gamma_o &= a_{\gamma_o} \\
m_{o_i} &\sim Uniform(0.5m_{o_i}^{(e)}, 2.0m_{o_i}^{(e)}) \\
\delta_{o_i} &\overset{iid}{\sim} N(\theta_o, \xi_o^2) \\
\theta_o &\sim N(m_{\theta_o}, s_{\theta_o}^2) \\
\xi_o^2 &\sim IG(a_{\xi_o^2}, b_{\xi_o^2}) \\
k_{o_i} &\overset{iid}{\sim} G(\phi_o, \omega_o) \\
\phi_o &\sim G(a_{\phi_o}, b_{\phi_o}) \\
\omega_o &\sim G(a_{\omega_o}, b_{\omega_o}),
\end{aligned}$$

where a and b subscripted above indicate constants, and $G(a, b)$ indicates a gamma distribution with mean ab and variance ab^2. The highly parametric specification above suggest that sensitivity may result from choices of distributional form as well as hyperparameter choices. As with any analysis, increasing the degree of assumption increases the *potential* for sensitivity to those assumptions. For example, inadequate sample sizes will certainly exacerbate these sensitivities. We consider a sensitivity study in Section 4 to examine the degree to which our results depend on the hyperparameter choices given above.

There are similarities between this approach to the quantification of expert opinion and Zellner's approach using g-prior distributions (Zellner 1986; Agliari and Parisetti 1988). Both approaches rely on the natural conjugate prior for $(\underline{\beta}, \sigma^2)$. However, Zellner (1986) elicits posterior means for $\underline{\beta}$ and σ^2, while we elicit predicted observations $\underline{y}_o$. Agliari and Parisetti (1988) extend Zellner's methods to include a different design matrix XA; similarly, we do not require

that the factor levels where the expert elicitation occurs correspond to the levels where the physical or computer experimental data are collected.

2.3 Computer Experimental Data

We have used the expert opinion data to develop Stage 1 posterior distributions for $\underline{\beta}$ and σ^2. We continue to update our knowledge about these parameters using data from computer experiments. Let the computer data and associated model parameters be indexed by c where the j^{th} element of the response vector $\underline{Y}_c$ is y_{c_j}. Consider the following model:

$$\begin{aligned} \underline{Y}_c &\sim N(\mathbf{X}_c\underline{\beta} + \underline{\delta}_c, \sigma^2\mathbf{\Sigma}_c) \\ \underline{\beta}|\sigma^2 &\sim N(\underline{\mu}_c, \sigma^2\mathbf{C}_c) \\ \sigma^2 &\sim IG(\alpha_c, \gamma_c). \end{aligned}$$

For this development, assume that $\mathbf{\Sigma}_c$ and $\mathbf{C}_c$ have the same diagonal form as $\mathbf{\Sigma}_o$ and $\mathbf{C}_o$. The "prior" distributions for $\underline{\beta}|\sigma^2$ and σ^2 are the Stage 1 posterior distributions given the expert opinion data. The only other unspecified prior distributions are:

$$\begin{aligned} \delta_{c_j} &\overset{iid}{\sim} N(\theta_c, \xi_c^2) \\ \theta_c &\sim N(m_{\theta_c}, s_{\theta_c}^2) \\ \xi_c^2 &\sim IG(a_{\xi_c^2}, b_{\xi_c^2}) \\ k_{c_j} &\overset{iid}{\sim} G(\phi_c, \omega_c) \\ \phi_c &\sim G(a_{\phi_c}, b_{\phi_c}) \\ \omega_c &\sim G(a_{\omega_c}, b_{\omega_c}). \end{aligned}$$

Although assuming a diagonal structure for $\mathbf{\Sigma}_c$ yields a model for the computer experiment where the observations are *conditionally* independent given $\underline{\beta}$, $\underline{\delta}_c$, σ^2, and $\underline{k}_c$, the observations are not *unconditionally* independent once the uncertainty in the unknown parameters is integrated out. For example, Broemeling (1985) derives the distribution for $\underline{Y}_c$ for the conjugate Bayesian linear model. Our model for the correlation structure differs from those proposed in Currin et al.

(1991) and Welch et al. (1992), where a distance-based parametric form for $\mathbf{\Sigma}_c$ is assumed with the parameters selected using cross-validation or maximum likelihood estimation. Although these forms of prior distribution could be incorporated into our analysis, we have chosen to induce correlation through the hierarchical structure of the prior.

Computer models, especially when the physical processes are not well known, often produce estimates that are biased with respect to the physical data. These biases may be in the mean structure (location bias) or in the variance (scale bias). Computer experimental data are especially likely to have scale biases, as these data usually tend to be less variable than physical experimental data; in fact, there is often no stochastic variability for given values of the factors, because a computer code is often deterministic. The variability occurs relative to the assumed model. Another reason for the reduced variability relative to physical experimental data is that we know that not all factors generating the physical experimental data are incorporated into the computer code—perhaps all of the factors causing variability are unknown. Since the location bias addresses only differences in the intercept term (β_0) between the computer and physical data, more general bias structures for the parameters can also be modeled. In Section 3, we motivate these ideas by introducing the operation of fluidized beds and the computer models for that process.

Because the location biases are additive (instead of multiplicative) the model only requires that data exist for a subset of the full set of factors. That is, if only one data source includes information on a factor, then only that source is used in estimating that effect. The precision with which those effects are estimated will be affected by the different amount of data used in estimation. However, distributions can be calculated. If model choice is to be done using the physical data only (as it is in our example), then all the factors would need to be present in the physical experimental data. Thus, the framework is quite general and does not require that all factors be present in each data source.

Other approaches that might be considered for modeling the computer experimental data often employ a Gaussian process (GP) model (REFERENCE HERE). While the GP approach

is commonly (and appropriately) used for many problems, the RBHM provides an alternative that is useful and easily interpreted for a large class of problems. We will illustrate a problem that is well suited for our proposed modeling approach in Section 3.

2.4 Incorporating Physical Experimental Data

Recall from Section 2.1 that the model for the physical experimental data is:

$$\underline{Y}_p \sim N(\mathbf{X}_p\underline{\beta}, \sigma^2 I).$$

After incorporating the computer experimental data into the analysis, we have a Stage 2 posterior that is used as the prior for $(\underline{\beta}, \sigma^2)$ in the Stage 3 analysis.

The Stage 3 analysis calculates the final distributions for the parameters of interest. These calculations cannot be done in closed form, but are carried out using Markov Chain Monte Carlo (MCMC). See the Appendix for general information on MCMC and the Metropolis-Hastings algorithm.

3 Application of RBHMs to Fluidized Bed Processes

Fluidized bed microencapsulation processes are used in the food industry to coat certain food products with additives. Dewettinck et al. (1999) describe a physical experiment and several corresponding thermodynamic computer models that were developed for predicting the steady-state thermodynamic operation point of a Glatt GPCG-1 fluidized bed unit in the top-spray configuration. Figure 1 illustrates the simple geometry of this unit, which is essentially an upside-down, truncated cone. The base of the unit contains a screen, below which there is an air pump. Also, there are coating sprayers at the side of the unit.

[Figure 1 about here]

To use the unit, a batch of uncoated food product is placed inside the "cone", and the air pump and coating sprayers are turned on. This "fluidizes" the product in the unit and coats the product as it passes by the sprayer. This is continued until a desired coating thickness is

achieved.

When room conditions and process conditions are constant, a fluidized bed process will attain its steady-state thermodynamic operation point. This state can be described in terms of the temperature and humidity inside the unit. The importance of the steady-state operation point is that product characteristics, such as coating evenness and efficiency, are directly related to it.

Several variables potentially affect the steady-state thermodynamic operating point; namely,

- V_f, fluid velocity of the fluidization air
- T_a, temperature of the air from the pump
- R_f, flow rate of the coating solution
- T_s, temperature of the coating solution
- M_d, coating solution dry matter content
- P_a, pressure of atomization air.

The ambient room conditions inside the plant, such as temperature T_r and humidity H_r, may also have an effect on the steady-state process conditions.

3.1 The Data

Dewettinck et al. (1999) consider twenty-eight process conditions of particular interest (settings) for a GPCG-1 fluidized bed process. In the experiment, distilled water was used as the coating solution. Thus, M_d was 0 (no dry matter content) for all 28 runs. Also, T_s was at room temperature (about 20^o C) for all 28 runs. Table 1 shows the room conditions (i.e., T_r and H_r) and settings for the remaining four process variables (i.e., T_a, R_f, P_a, and V_f).

For factor combination, glass beads were put in the unit, and the process was run for 15 minutes to attain steady-state. Then, temperature inside the unit was measured at 20, 25, and 30 minutes, and their average was recorded. The average outlet air temperature (the steady-state response of interest), $T_{2,exp}$, is reported in Table 2. Also, three unique computer models were

also considered by Dewettinck et al. (1999) to predict the steady-state outlet air temperature for each run. These computational responses are also given in Table 2 and are labeled as $T_{2,1}$, $T_{2,2}$, and $T_{2,3}$, respectively.

There are important differences among the three computational models that are described in detail in Dewettinck et al. (1999). In summary, the first computer model does not include adjustments for heat losses in the process. The second computer model takes those heat losses into account. A further adjustment for the inlet airflow represents the fundamental difference between the second and third computer models.

[Table 1 about here]

[Table 2 about here]

3.2 Modeling $T_{2,exp}$ in Terms of Room and Process Conditions

Table 3 shows the correlation matrix for the room conditions, process conditions, and observed steady-state temperature $T_{2,exp}$. Figure 2 is a matrix plot of these seven variables. Note that T_a has the highest correlation with $T_{2,exp}$ ($r = 0.73$).

[Table 3 about here]

Choice of a model is complicated by the fact that the underlying design is not at all clear. The covariance matrix reveals that some of the covariates are highly correlated (as high as 0.82) indicating possible collinearity. We also note that the full second-order model is fully saturated.

[Figure 2 about here]

Chipman, Hamada, and Wu (1997) describe a Bayesian variable selection procedure that places hierarchical prior distributions on second-order effects. In their approach, higher prior probability is given to interactions if one of the main effects is in the model, and an even higher probability is placed on interactions when both main effects are in the model. Using their approach on the physical data, we obtain the variable selection results displayed in Table 4, which provide the most likely models and their respective posterior probabilities.

To illustrate the RBHM approach, we use the most likely model from Table 4 to form $\mathbf{X}\underline{\beta}$,

where $\mathbf{X}$ is composed of a column of ones (for the intercept) and columns corresponding to T_a, R_f, V_f, and $R_f \times V_f$, whose respective regression parameters are $\underline{\beta} = (\beta_0, \ldots, \beta_4)'$.

[Table 4 about here]

Table 5 contains the OLS fit of the most likely model in Table 4.

[Table 5 about here]

In our example the hyperparameter values are given in Table 6. Note that the same hyperparameters were used for all three computer experiments. As we have no prior knowledge as to the sign of the location bias, we center the distribution of δ_{c_i} at zero (that is, unbiased in location), and allow the mean of that distribution to have a standard deviation of 10. While we believe the computer models are all reasonably good approximations of the physical model, we do not have a good idea about the degree of separation, and thus allow a generous variability for the location biases ($a_{\xi_c^2} = 2000$ and $b_{\xi_c^2} = 3.0$ suggest a mean for the variance distribution of $2000/(3-1) = 1000$ and a standard deviation of $2000/((3-1)*(3-2)) = 1000$). The distribution of scale biases is also somewhat unknown. With little or no prior knowledge, we would allow the mean of the scale biases to be unity (unbiased in scale). Further, we believe the standard deviation of the scale biases should be no greater than 15, thus we let the mean of the scale bias distribution be 1 and the standard deviation 15. This allows a generous range for the scale biases.

[Table 6 about here]

3.3 RBHM Analysis Results

Figures 3*(a)-(e)* show the posterior for $\underline{\beta}$ with only the physical experimental data, the physical with each of the computer experimental data taken separately, and the final posterior distribution for $\underline{\beta}$ after incorporating all sources of information. Figure 3*(f)* shows the corresponding posteriors for σ^2. The figures indicate two important and appealing aspects of our RBHM approach. First, the additional sources of information reduce uncertainty in the distribution of the parameters, thus making our estimates more precise. Second, the additional data sources

do not necessarily contain the same amount of information (although, in our example, they do have the same number of observations).

In addition to posterior distributions for $\underline{\beta}$ and σ^2, our modeling approach allows us to estimate the bias terms. As an illustration, Figures 4*(a)* and 4*(b)* present the location and scale bias predictive distributions for each of the computer models. Note that these distributions are integrated over the distribution of individual specific location and scale bias terms. One appealing feature of these plots is that they indicate a new approach to computer model validation, relative to the physical observations. Those models that have most mass over 0 are less location biased for the physical experimental data. For example, the bias is more concentrated around 0 for the third computer model than for the other two computer models. These plots also reveal the uncertainty associated with the bias terms (a feature that cannot easily be inferred from a casual examination of the data). Note that the third model is the computer model that attempts to account for more phenomena. Figure 4*(b)* reveals that all three computer models tend to underestimate the variability in the physical experimental data. Scale bias terms more than one (because the scale bias is parameterized as $1/k_{C_i}$) indicate underestimation of variability.

[Figure 3-4 about here]

Table 7 contains the maximum likelihood estimate (MLE), 95% confidence intervals (calculated from only the physical experimental data), the posterior mean and 95% highest posterior density (HPD) intervals calculated using the integrated computer and physical experimental data for $\underline{\beta}$ and σ^2. Recall that an HPD interval is the shortest interval in the posterior distribution containing 95% of the posterior probability. Notice that the HPD intervals are shorter, sometimes significantly so, than the 95% confidence intervals, reflecting the additional information that has been incorporated into the analysis.

[Table 7 about here]

3.3.1 Expert Opinion Data

Although no expert opinions were available for use in the fluidized bed example, it is interesting to observe the impact of such data on the results. For purely illustrative purposes, suppose that eight expert opinions were elicited for use in the fluidized bed example.

[Table 8 about here]

The expert opinions are shown in Table 8 where $T_{2,o}$ denotes the expected steady-state outlet air temperature, $q_{0.9}$ is the corresponding subjective 0.9 quantile on the outlet air temperature, and $m_o^{(e)}$ is the equivalent "worth" of the opinion (see Section 2.2).

[Figure 5 about here]

Figure 5 contains two posterior distributions, one for the regression coefficient for flow rate (β_2) and one for the error variance (σ^2). The solid line is the posterior distribution conditional on the artificial expert opinion with one computer model and the physical experimental data. The dotted line is the posterior distribution with only the physical experimental data and one computer model. Due to estimation of location and scale biases for both the computer data and the artificial expert opinion data, there is only a small gain in information for adding the expert opinion data. No inference from these posterior distributions should be made as the expert opinion data was generated for illustration purposes only.

4 Prior Sensitivity

Bayesian analyses which contain many parameters have the potential to rely heavily on prior distributions and prior parameter choice. To assess the impact of our choices of prior parameters (and hyperparameters) we present a small sensitivity analysis in this section.

To address hyperparameter sensitivity, we designed a 2^{10-5} fractional factorial design where we chose "high" and "low" values that we deemed feasible. The values we chose are shown in Table 9.

[Table 9 about here]

Marginal posterior distributions for the regression coefficients ($\beta_0, \ldots, \beta_4$) and the error vari-

ance (σ^2) are shown in Figure 6. Because the posterior distributions do not lie exactly on one another, there is clearly some sensitivity to prior specification. The differences in the posterior distributions are not significant however. The only clear deviation is 6 of the 32 fractional factorial combinations for σ^2, and these produce significant departure from the posterior distributions presented in Section 3.3. We note that each of these stems from a prior distribution which includes nearly no mass around the posterior distribution. That is, they represent infeasible prior distributions. This indicates that care should be taken when specifying prior parameters on the variability. Sensitivity is only observed when priors are completely misspecified.

[Figure 6 about here]

5 Conclusions/Discussion

When expert opinion is elicited, an equivalent number of observations $m_{o_i}^{(e)}$ is also stated that reflects its worth in terms of a number of equivalent physical observations. This parameter is not required for the computer experimental data because this information is captured in the prior parameters $\theta_c, \xi_c^2, \phi_c$, and ω_c. These parameters control the prior information about the location and scale biases for the computer experimental data. If the biases are known exactly (a point mass prior), then each computer observation counts as exactly one physical observation—no information must be used to estimate the biases, and it can all be used to estimate $\underline{\beta}$ and σ^2. If these parameters are used to specify a very diffuse ("non-informative") prior with close to infinite variances, then each computer observation counts for only a tiny fraction of a physical observation. If the parameters specify an informative prior, then the computer observations account for some intermediate fraction of a physical observation.

The model that we have used in our example treats each computer model independently. In the extreme, this implies that if the three models were identical, we would count each observation three times the fraction of a physical observation implied by the prior distributions. We can change this by modeling correlations between the computer models. There are two obvious ways to do this. The simplest is to add a hierarchical structure on the hyperparameters (θ_c, ξ_c, ϕ_c,

and ω_c) of the various computer models. As discussed in Section 2.3, this induces correlations in the unconditional distributions of the computer observations. A second is to model the entire vector of observations from the three computer models directly as a multivariate normal and to specify an appropriate covariance structure. This choice would be especially appropriate in the case where we had precise information about the differences in the physics modeled by the individual computer models. For this example, we have insufficient knowledge about the precise similarities/differences between the three computer models to permit either of these to be used.

We have not imposed the requirement that the computed (or measured) value of the response variable be considered at the same factor values in both experiments. We only require that there exist some common set of factors (either all or at least some) for both experiments. While the example does not fully illustrate this, it is an important feature in the general model. As the analysis proceeds by using information from one type of experiment to update the distribution of the parameters, if there is no data at a particular design point for a particular experiment, then the distribution for the parameter remains unchanged, except for correlations which may exist in the parameters.

As with any Bayesian analysis, there is sensitivity to the specification of the prior distributions for the hyperparameters. Fortunately, however, the sensitivity is only particularly acute when the priors are completely misspecified. Although some of the hyperparameter selections in Section 3.2 are somewhat arbitrary, they illustrate the kinds of discussions that the analyst would engage in with the data owner to come to "reasonable" hyperparameter distributions. If at all possible, we prefer diffuse but informative prior distributions using expert input.

In this example, we included all three sets of computer data, even though we believed that the models were successively improved. There are two reasons for this choice: first, we believe that by appropriate modeling of biases, there is information in all of the codes that should not be discarded, and second, it is often of interest to characterize the biases of each code relative to the physical data.

We have presented an RBHM that can be used to combine data from both computer and

physical experiments. When available, expert opinion data is also used to "sharpen" the initial informative, but rather diffuse, prior distributions. Appropriate biases are introduced as a way to account for differences in these data sources. Sample results indicate that significantly more precise estimates of the regression coefficients and error variance are obtained by means of this method. In addition, the methodology can be used to recursively estimate those unknown biases of particular interest. Biases that are not particularly interesting can be marginalized (this is, averaged out of the analysis using appropriate priors). Obviously, not all problems involving combination of computer models and physical experiments are well suited to combination through statistical (response surface) models. In our example, however, the approach is well suited to the data collected and the biases seem to reflect the actual differences between the computer models and the physical data.

The methodology can also be used to combine various other kinds of experimental information. Similarly, information from more than two physical and/or computer experiments can also be combined using the RBHM simply by considering an appropriate bias structure for each data source and by increasing the number of stages in the analysis accordingly.

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A Appendix

A.1 Markov Chain Monte Carlo (MCMC)

Suppose we are interested in making statistical inference about a parameter (possibly vector valued) Θ. We characterize our information (or lack of information) about the distribution of

$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ as $\pi(\Theta)$ (prior distribution). Data are collected and represented by the likelihood or $f(\mathbf{x}|\Theta)$. In any Bayesian analysis, inference on the parameters depends on the calculated posterior distribution

$$\pi(\Theta|\mathbf{x}) = \frac{\pi(\Theta)f(\mathbf{x}|\Theta)}{\int_\Theta \pi(\Theta)f(\mathbf{x}|\Theta)d\Theta}. \tag{2}$$

In many situations, the denominator of (2) is not a well known integral and must be calculated numerically; e.g., Markov Chain Monte Carlo (MCMC). Let Θ_{-v} be Θ with the v^{th} element removed. A successive substitution implementation of the MCMC algorithm proceeds as follows:

(1) Initialize $\Theta^{(0)}$ and set $t = 1$.

(2) Set $v = 1$.

(3) Generate an observation $\theta_v^{(t)}$ from the distribution of $[\theta_v|\Theta_{-v}^{(t-1)}]$, replacing recently generated elements of $\Theta_{-v}^{(t-1)}$ with elements of $\Theta_{-v}^{(t)}$ if they have been generated.

(4) Increment v by 1 and go to (3) until $v = n$.

(5) If $v = n$ increment t by 1 go to (2).

under conditions outlined in Hastings (1970), as $t \to \infty$ the distribution of $\{\theta_1^{(t)}, \ldots, \theta_n^{(t)}\}$ tends to the joint posterior distribution of Θ, as desired.

Typical implementation of the algorithm generates an initial "large" number of iterations (called the *burn-in*) until the observations have converged. The burn-in samples are discarded, and the observations generated thereafter are used as observations from the posterior distribution of Θ. Nonparametric density estimators (Silverman 1986) can then be used to approximate the posterior distribution.

A.2 Metropolis-Hastings

Some complete conditional distributions may not be available in closed form. That is, it may be difficult to sample from $[\theta_v|\Theta_{-v}^{(t-1)}] \propto g(\theta_v)$. Obtaining observations from such distributions

is facilitated by implementing a *Metropolis-Hastings* step (Hastings 1970) for step (3) in the algorithm above. This is difficult because the distribution is known only up to a constant.

(1) Initialize $\theta_{v_{old}}^{(0)}$ and set $j = 0$.

(2) Generate an observation $\theta_{v_{new}}^{(j)}$ from a *candidate* distribution $q(\theta_{v_{old}}^{(j)}, \theta_{v_{new}}^{(j)})$, where $q(x, y)$ is a probability density in y with mean x.

(3) Generate a uniform (0,1) observation u.

(4) Let

$$\theta_{v_{new}}^{(j+1)} = \begin{cases} \theta_{v_{new}}^{(j)}, & \text{if } u \leq \alpha(\theta_{v_{old}}^{(j)}, \theta_{v_{new}}^{(j)}) \\ \theta_{v_{old}}^{(j)}, & \text{otherwise,} \end{cases}$$

where $\alpha(x, y) = \min\left\{\frac{g(y)q(y,x)}{g(x)q(x,y)}, 1\right\}$.

(5) Increment j and go to (2).

The candidate distribution can be almost any distribution (Gilks et al. 1996), although a symmetric distribution such as the normal results in a simplification of the algorithm, and is called a *Metropolis step* (as opposed to a Metropolis-Hastings step). A common choice for $q(x, y)$ is a normal distribution with mean x and some variance which allows the random deviates to be a representative sample from the entire complete conditional distribution. A rule of thumb given in Gilks et al. (1996) suggests that the variance in $q(x, y)$ be one-third of the sample variance of the observed data.

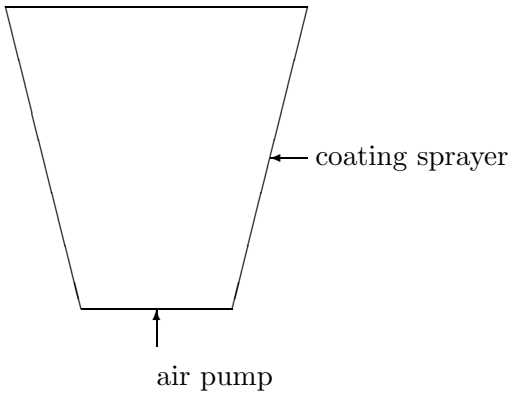


Figure 1: A Glatt GPCG-1 fluidized bed unit.

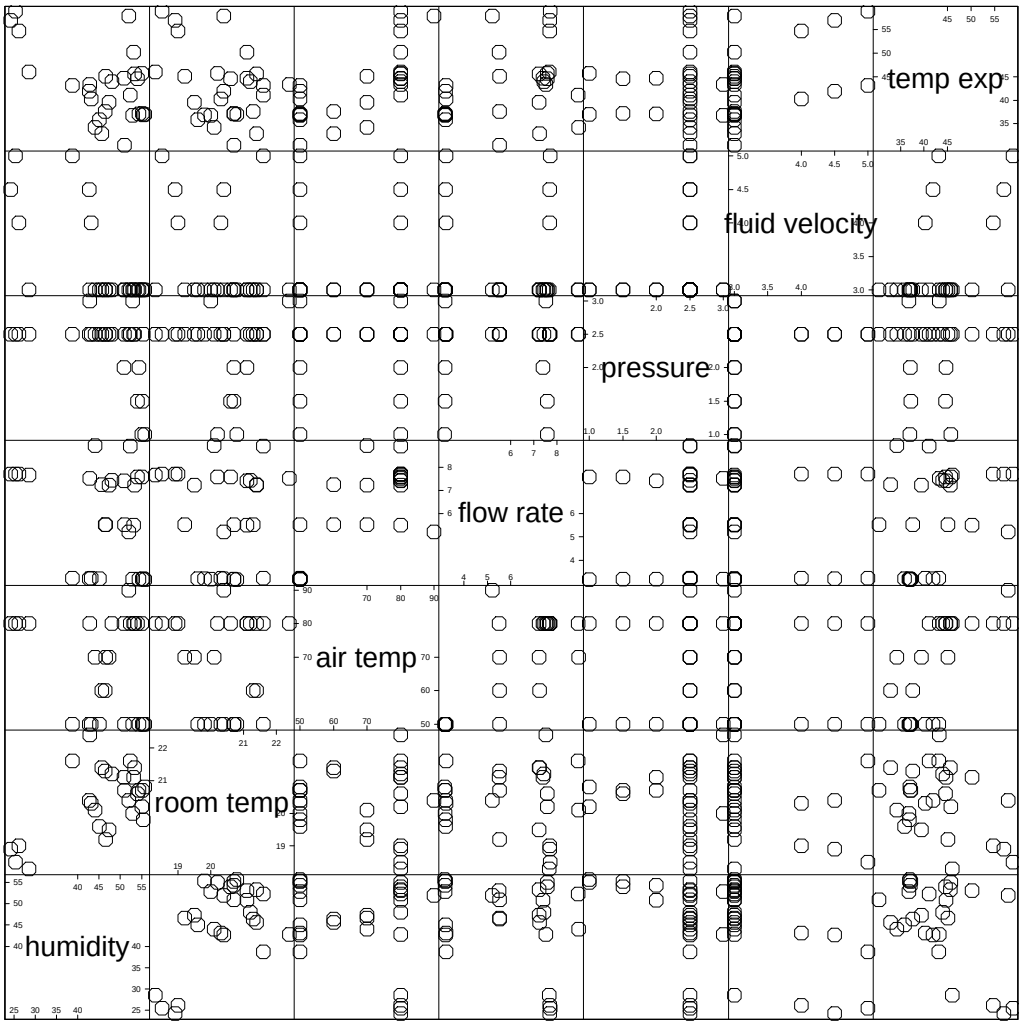


Figure 2: Scatterplot matrix of the experimental response with each of the 6 covariates.

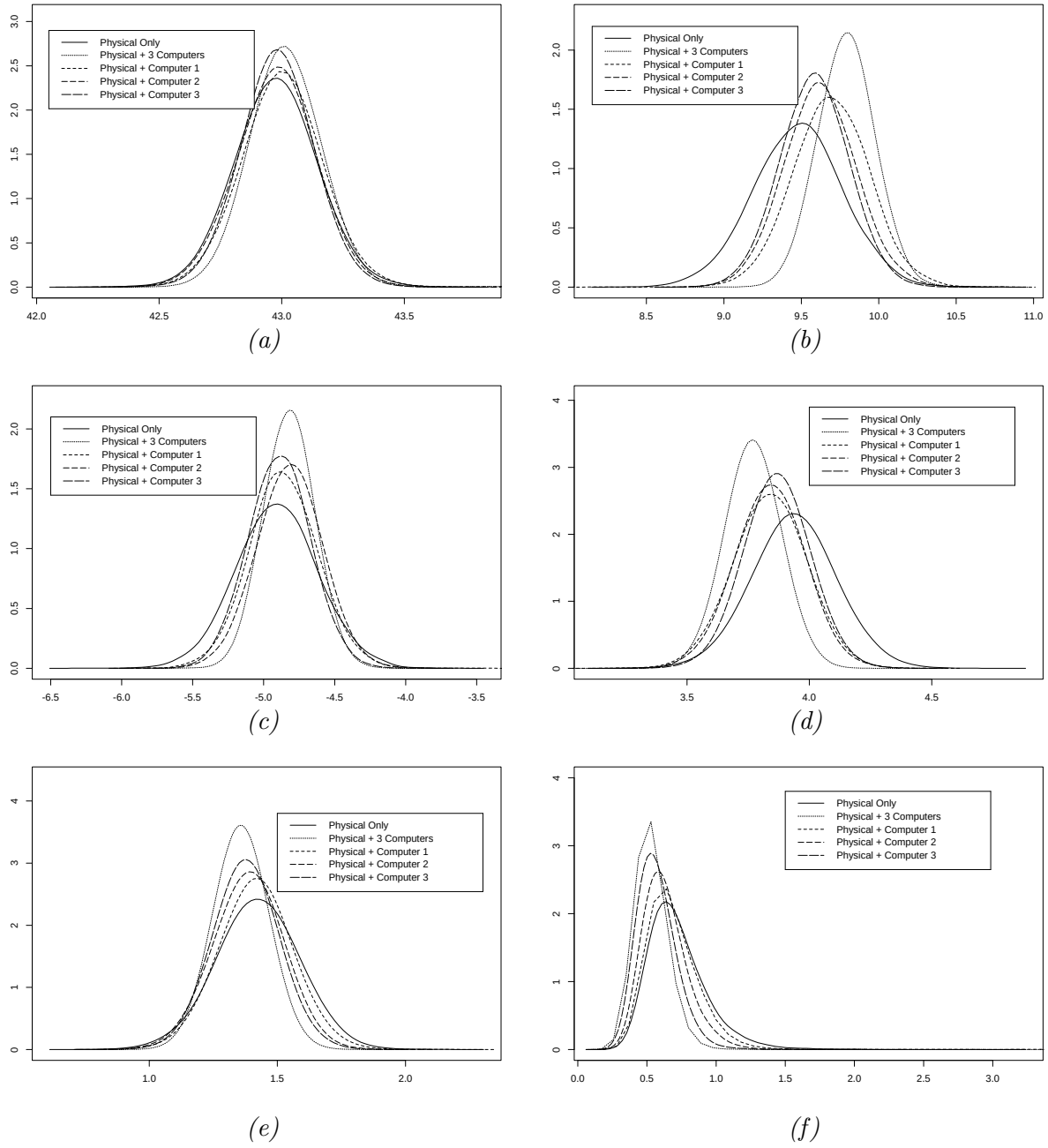


Figure 3: Comparison of posterior distributions conditional on different sources of information: *(a)* β_0, intercept; *(b)* β_1, air temperature; *(c)* β_2, flow rate; *(d)* β_3, fluid velocity; *(e)* β_4, interaction between flow rate and fluid velocity; and *(f)* σ^2. The different lines indicate inclusion of different data sources.

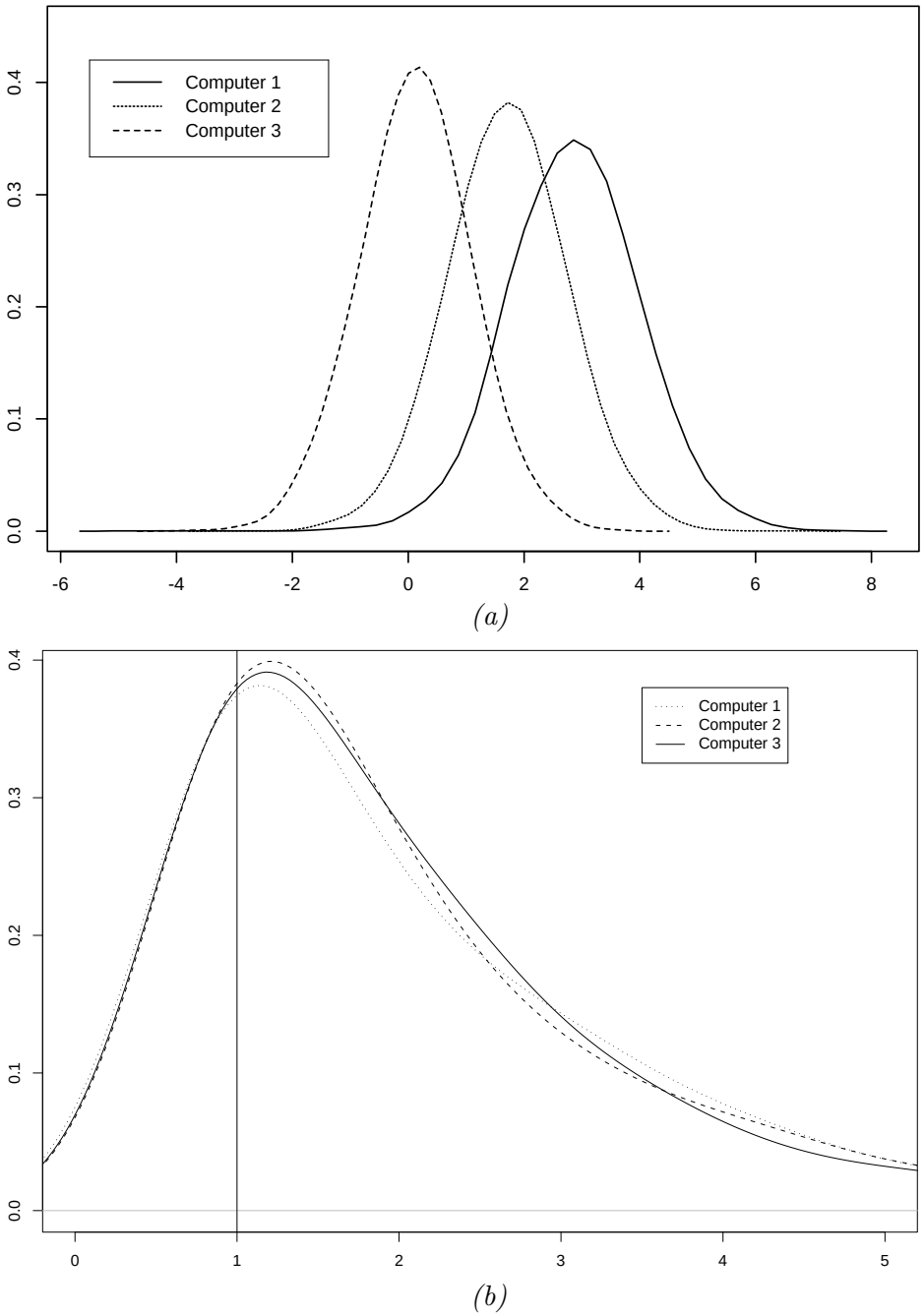
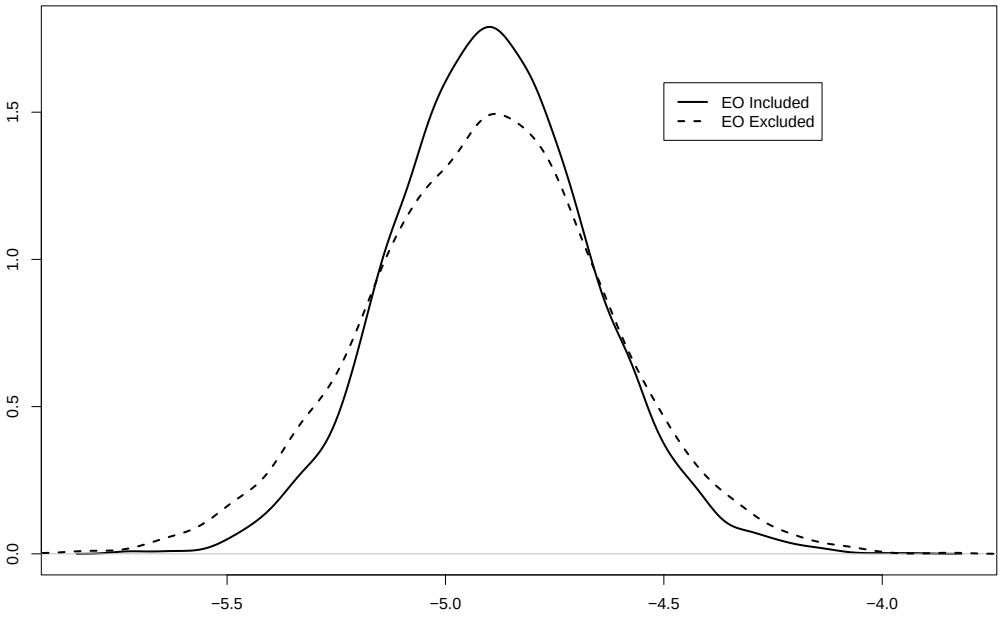
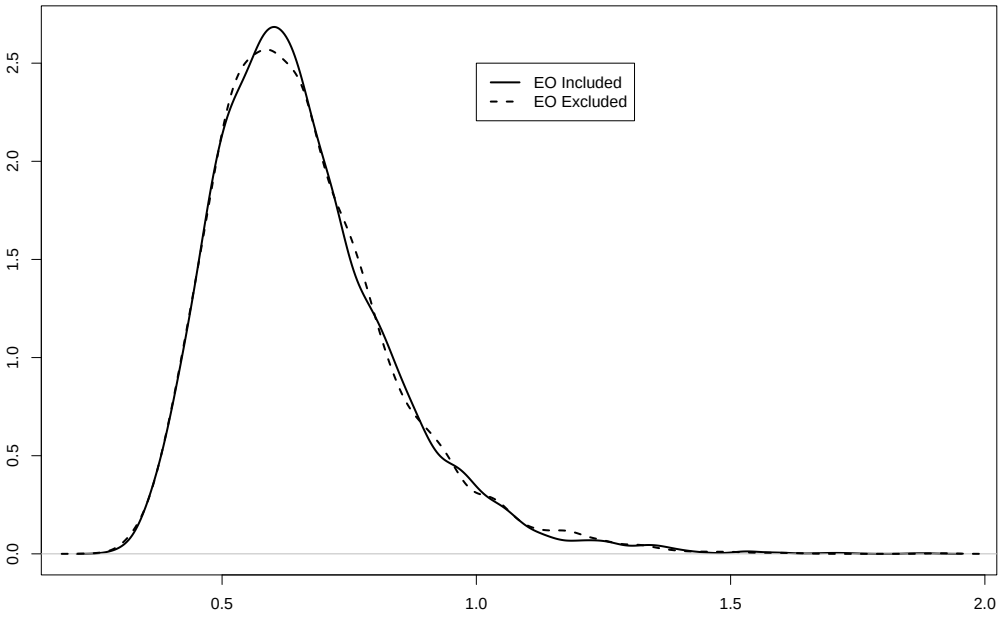


Figure 4: Comparison of *(a)* location bias and *(b)* scale bias predictive distributions for three different computer models of the fluidized bed process.



(a)



(b)

Figure 5: Comparison of posterior distributions for *(a)* β_2, flow rate and *(f)* σ^2. The solid line is the posterior distribution conditional on the artificial expert opinion with one computer model and the physical experimental data. The dotted line is the posterior distribution with only the physical experimental data and one computer model.

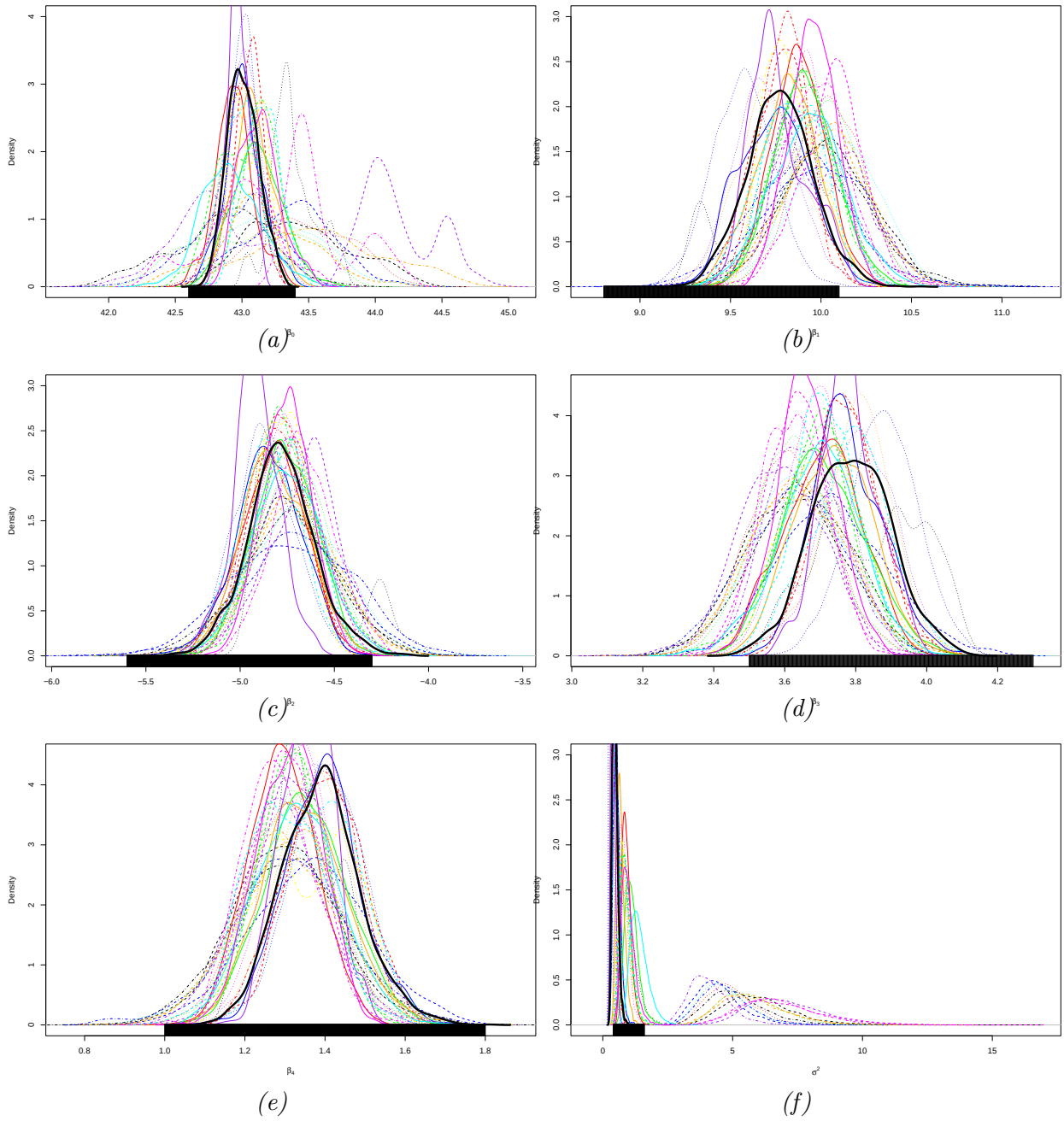


Figure 6: Sensitivity analysis for selected hyperparameters in the RBHM formulation: *(a)* β_0, intercept; *(b)* β_1, air temperature; *(c)* β_2, flow rate; *(d)* β_3, fluid velocity; *(e)* β_4, interaction between flow rate and fluid velocity; and *(f)* σ^2. The different lines indicate a different factorial combination in the sensitivity analysis, and the thick solid line indicates the posterior at the original settings. The "rug" at the bottom of each picture is the frequentist confidence interval based on the physical data only fit to the five parameter linear model.

Table 1: Process Variables

H_r (%)	T_r ($^\circ C$)	T_a ($^\circ C$)	R_f (g/min)	P_a (bar)	V_f (m/s)
51.0	20.7	50	5.52	2.5	3.0
46.4	21.3	60	5.53	2.5	3.0
46.6	19.2	70	5.53	2.5	3.0
53.1	21.1	80	5.51	2.5	3.0
52.0	20.4	90	5.21	2.5	3.0
45.6	21.4	60	7.25	2.5	3.0
47.3	19.5	70	7.23	2.5	3.0
53.3	21.4	80	7.23	2.5	3.0
44.0	20.1	70	8.93	2.5	3.0
52.3	21.6	80	8.91	2.5	3.0
55.0	20.2	80	7.57	1.0	3.0
54.0	20.6	80	7.58	1.5	3.0
50.8	21.1	80	7.40	2.0	3.0
48.0	21.2	80	7.43	2.5	3.0
42.8	22.4	80	7.51	3.0	3.0
55.7	20.8	50	3.17	1.0	3.0
55.2	20.7	50	3.18	1.5	3.0
54.4	20.7	50	3.19	2.0	3.0
55.4	19.8	50	3.20	2.5	3.0
52.9	20.0	50	3.19	3.0	3.0
28.5	18.3	80	7.66	2.5	3.0
26.1	19.0	80	7.69	2.5	4.0
24.2	18.9	80	7.69	2.5	4.5
25.4	18.5	80	7.70	2.5	5.0
45.1	19.6	50	3.20	2.5	3.0
43.1	20.3	50	3.23	2.5	4.0
42.7	20.4	50	3.20	2.5	4.5
38.7	21.6	50	3.22	2.5	5.0

Table 2: Experimental and Computer Model Steady-State Temperatures

$T_{2,exp}$ (°C)	$T_{2,1}$ (°C)	$T_{2,2}$ (°C)	$T_{2,3}$ (°C)
30.4	32.4	31.5	30.2
37.6	39.5	38.5	37.0
45.1	46.8	45.5	43.7
50.2	53.8	52.6	51.0
57.9	61.7	59.9	58.2
32.9	35.2	34.6	32.6
39.5	42.4	41.0	39.1
45.6	49.5	48.5	46.4
34.2	37.5	36.6	34.8
41.1	45.5	44.3	42.0
45.7	50.5	49.0	47.0
44.6	49.8	48.4	46.3
44.7	49.8	48.4	46.3
44.0	49.2	48.0	45.7
43.3	48.6	47.5	45.4
37.0	39.5	38.0	37.7
37.2	39.5	38.5	37.1
37.1	39.5	37.5	36.7
36.9	39.5	38.5	36.1
36.8	37.7	37.2	36.2
46.0	48.7	47.3	45.1
54.7	57.7	56.2	54.2
57.0	60.1	58.7	57.0
58.9	62.0	60.5	58.7
35.9	37.9	37.1	36.1
40.3	41.7	40.8	40.1
41.9	43.0	42.3	41.4
43.1	43.9	43.3	42.6

Table 3: Correlation Matrix

	H_r	T_r	T_a	R_f	P_a	V_f	$T_{2,exp}$
H_r	1.00	0.57	−0.26	−0.33	−0.39	−0.69	−0.53
T_r	0.57	1.00	−0.09	−0.07	−0.04	−0.28	−0.37
T_a	−0.26	−0.09	1.00	0.82	0.06	−0.08	0.73
R_f	−0.33	−0.07	0.82	1.00	0.09	−0.10	0.35
P_a	−0.39	−0.04	0.06	0.09	1.00	0.18	0.08
V_f	−0.69	−0.28	−0.08	−0.10	0.18	1.00	0.47
$T_{2,exp}$	−0.53	−0.37	0.73	0.35	0.08	0.47	1.00

Table 4: Bayesian Variable Selection Results

Model	Pr(Model|Data)
$\beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 R_f \times V_f$	0.1169
$\beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 H_r \times T_r + \beta_5 R_f \times V_f$	0.0349
$\beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 H_r^2$	0.0155
$\beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 T_r \times T_a + \beta_5 R_f \times V_f$	0.0141
$\beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 R_f \times V_f + \beta_5 V_f^2$	0.0136
$\beta_0 + \beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 H_r^2$	0.0132
$\beta_1 T_r + \beta_2 T_a + \beta_3 R_f + \beta_4 V_f + \beta_5 R_f \times V_f$	0.0130

Table 5: OLS fit for $T_{2,exp} = \beta_0 + \beta_1 T_a + \beta_2 R_f + \beta_3 V_f + \beta_4 R_f \times V_f + \epsilon$

Variable	DF	Parameter Estimate	Standard Error	T for $H_\circ$: Parameter= 0	Prob> $\|T\|$
Intercept	1	42.9769	0.1714	250.7352	0.0000
T_a	1	9.4756	0.3056	31.0076	0.0000
R_f	1	-4.9048	0.3035	-16.1626	0.0000
V_f	1	3.9345	0.1761	22.3445	0.0000
$R_f \times V_f$	1	1.4263	0.1671	8.5336	0.0000

Table 6: Hyperparameter Values for Parameters in Computer Experiments

Hyperparameter	Value
C_c	1.0×10^{-4}
α_C	3.0
β_C	3.0
m_{θ_c}	0.0
$s^2_{\theta_c}$	100.0
$a_{\xi_c^2}$	2000.0
$b_{\xi_c^2}$	3.0
a_{ϕ_c}	1.0×10^{-3}
b_{ϕ_c}	1.0×10^{-3}
a_{ω_c}	1.0×10^{-3}
b_{ω_c}	1.0×10^{-3}

Table 7: Comparison of Confidence and Credible Intervals

		95% Confidence Int.			95% Credible HPD Int.	
	MLE	Lower	Upper	Post. Mean	Lower	Upper
σ^2	0.81	0.49	1.60	0.53	0.36	0.77
β_0	42.97	42.62	43.33	43.01	42.75	43.28
β_1	9.47	8.84	10.10	9.79	9.44	10.13
β_2	-4.90	-5.53	-4.27	-4.82	-5.15	-4.48
β_3	3.93	3.57	4.29	3.76	3.56	3.96
β_4	1.42	1.08	1.77	1.35	1.17	1.53

Table 8: Example Expert Opinion Data

T_a $(^\circ C)$	R_f (g/min)	V_f (m/s)	$T_{2,o}$ $(^\circ C)$	$q_{0.9}$ $(^\circ C)$	$m_o^{(e)}$
50	3	3	37	39	0.5
90	3	3	68	70	0.5
50	9	3	23	25	0.5
90	9	3	51	53	0.75
50	3	5	49	53	1.0
90	3	5	75	77	0.5
50	9	5	42	43	0.75
90	9	5	69	72	0.5

Table 9: Hyperparameter Values for Sensitivity Analysis

Hyperparameter	Low	High
C_c	1.0×10^{-7}	0.1
α_C	0.1	5.0
β_C	0.1	5.0
$s^2_{\theta_c}$	50	500
$a_{\xi_c^2}$	100	1000
$b_{\xi_c^2}$	1	10
a_{ϕ_c}	1.0×10^{-2}	2
b_{ϕ_c}	1.0×10^{-2}	2
a_{ω_c}	1.0×10^{-2}	2
b_{ω_c}	1.0×10^{-2}	2